A VARIATIONAL APPROACH TO THE NONLINEAR HELMHOLTZ EQUATION WITH LOCAL NONLINEARITY

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Abstract. In this paper, we study real solutions of the nonlinear Helmholtz equation

$$-\Delta u - k^2 u = f(x,u), \qquad x \in \mathbb{R}^N$$

satisfying the asymptotic conditions

$$u(x) = O(|x|^{\frac{1-N}{2}}) \quad \text{and} \quad |x|^{\frac{N-1}{2}} \left(\frac{\partial^2 u}{\partial r^2}(x) + k^2 u(x) \right) \to 0 \qquad \text{as } r = |x| \to \infty.$$

We develop the variational framework to prove the existence of nontrivial solutions for compactly supported nonlinearities without any symmetry assumptions. In addition we consider the radial case, in which, for a larger class of nonlinearities, infinitely many solutions are shown to exist. Our results give rise to the existence of standing wave solutions of corresponding nonlinear Klein-Gordon equations with arbitrarily large frequency.

1. Introduction

The study of the existence and qualitative properties of solutions to nonlinear wave equations

$$\frac{\partial^2 \psi}{\partial t^2}(t,x) - \Delta \psi(t,x) + V(x)\psi(t,x) = f(x,\psi(t,x)), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1}$$

goes back to the sixties, see e.g. the classical paper by Jörgens [13]. Since then, many authors have been investigating various aspects of this problem, including the question of existence and orbital stability of *standing wave (or solitary wave) solutions*. These solutions are given by the ansatz

$$\psi(t,x) = e^{i\omega t} u(x), \qquad \omega \in \mathbb{R}. \tag{2}$$

Taking for example a nonlinearity of the form $f(x,\psi) = g(x,|\psi|^2)\psi$, we see that such a ψ solves (1) if and only if u solves the reduced wave equation

$$-\Delta u + V(x)u - \omega^2 u = f(x,u), \qquad x \in \mathbb{R}^N. \tag{3}$$

Note that, due to the presence of the linear potential V, it is natural to consider nonlinearities f satisfying $\partial_u f(x,0) = 0$ on $\mathbb{R}^N$. In this case, in almost all of the available literature it is assumed that ω^2 is not contained in the essential spectrum of the Schrödinger operator $-\Delta + V$. We refer the reader to the surveys and monographs [3, 8, 20, 25, 26, 30] and the references therein for results in this case. In the special case where $V \equiv V_0$ is a constant, this restriction amounts to assuming $V_0 > \omega^2 \geq 0$. Some authors have also considered the limiting case where ω^2 coincides with the infimum or another boundary point of the essential spectrum of V, see e.g. [15] for a survey of classical results and [1, 2, 3, 6, 9, 18, 22, 31] for more recent work in this case. On the contrary, very little seems to be known if ω^2 is

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contained in the interior of the essential spectrum of V. In the present paper, we consider this situation in the special case $V \equiv V_0$ and $\omega^2 > V_0$, so by setting $k^2 = \omega^2 - V_0$ we arrive at the nonlinear Helmholtz equation

$$-\Delta u - k^2 u = f(x,u), \qquad x \in \mathbb{R}^N, \tag{4}$$

with $k > 0$. It seems not clear a priori in which space one should approach this problem and if variational methods can be used. In the present paper, we provide first results in the case where f is supported in a bounded subset of $\mathbb{R}^N$, i.e. f vanishes on $[\mathbb{R}^N \setminus B_R] \times \mathbb{R}$ for $R > 0$ sufficiently large. Here $B_R \subset \mathbb{R}^N$ denotes the open ball centered at 0 with radius R. Note that in this case no solution of (4) exists in the spaces $L^2(\mathbb{R}^N)$ and $D^{1,2}(\mathbb{R}^N)$ (see e.g. [14]). Nevertheless, this restriction on f allows us to work with the Dirichlet to Neumann map for the exterior problem for the linear Helmholtz equation $\Delta u + k^2 u = 0$ on $\mathbb{R}^N \setminus B_R$ together with a suitable asymptotic condition on u. To explain this more precisely, let us suppose for a moment that the nonlinearity $f(x,u)$ is replaced by an inhomogeneous source term $f(x)$ supported in B_R. In this case, a well-studied problem is to analyze the far field expansion of the solution of (4) under the the Sommerfeld (or outgoing) radiation condition

$$r^{\frac{N-1}{2}} \left| \frac{\partial u}{\partial r}(x) - iku(x) \right| \to 0 \qquad \text{as } r = |x| \to \infty. \tag{5}$$

This condition has been introduced in Sommerfeld's classical work [23], where its physical significance is discussed in comparison with other asymptotic conditions. In this context, (5) is enforced to study outgoing waves excited by the source term $f(x)$. Moreover, by a well-known result (which goes back to Rellich [21]), for given sufficiently regular Dirichlet boundary data on $S_R := \partial B_R$ there exists a unique solution of $\Delta u + k^2 u = 0$ in $\mathbb{R}^N \setminus B_R$ satisfying (5). Furthermore, the corresponding Dirichlet to Neumann map T_R on S_R, also called the *capacity operator* (see [19]), is well understood and can be computed explicitly in terms of spherical harmonics, see Section 6 below. This operator assigns to a given boundary datum on S_R the normal derivative of the corresponding unique solution $\Delta u + k^2 u = 0$ in $\mathbb{R}^N \setminus B_R$ satisfying (5). In the present paper, we focus on standing wave solutions which — as explained in [23, pp. 328–329] — can be produced by superposing an outgoing wave and an incoming wave having opposite frequency. Technically, this will be realized by considering real functions and working with the real part K_R of T_R, which amounts to studying (4) together with the asymptotic conditions

$$u(x) = O(|x|^{\frac{1-N}{2}}) \quad \text{and} \quad |x|^{\frac{N-1}{2}} \left(\frac{\partial^2 u}{\partial r^2}(x) + k^2 u(x) \right) \to 0, \qquad \text{as } r = |x| \to \infty. \tag{6}$$

We note that on one hand, the restriction that f vanishes outside of B_R is convenient since it allows us to formulate problem (4), (6) as a variational problem in $H^1(B_R)$ using the operator K_R. On the other hand, we shall see that the vanishing of f will create additional difficulties in the proof of Cerami's condition which is needed to show the existence of critical points of the corresponding functional.

To state our results, we need to introduce further notation and to state our assumptions. Let 2^* denote the critical Sobolev exponent, i.e. $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := \infty$ if $N = 1, 2$. For our main result, we shall suppose that the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous, and that there exists a bounded set $\Omega \subset \mathbb{R}^N$ of positive measure such that the following holds:

(f_0) $f(x,u) = 0$ for $x \in \mathbb{R}^N \setminus \Omega$, $u \in \mathbb{R}$.

(f_1) There exists $a > 0$, $p \in (2, 2^*)$ such that $|f(x, u)| \le a(1 + |u|^{p-1})$ for every $x \in \mathbb{R}^N$, $u \in \mathbb{R}$.

(f_2) $f(x, u) = o(|u|)$ uniformly in x as $u \to 0$.

(f_3) $F(x, u) \ge 0$ for every $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, and $F(x, u)/u^2 \to \infty$ as $|u| \to \infty$ for every $x \in \Omega$.

(f_4) There exists $s_0 > 0$ such that for every $x \in \Omega$ we have $f(x, -s_0) \le 0 \le f(x, s_0)$, and the map $u \mapsto f(x, u)/|u|$ is nondecreasing on (s_0, ∞) and on $(-\infty, -s_0)$.

Here we set $F(x, u) = \int_0^u f(x, s)\, ds$ for $x \in \mathbb{R}^N$, $u \in \mathbb{R}$. We point out that nonlinearities of the type

$$f(x, u) = q(x)|u|^{p-2}u \qquad \text{or} \qquad f(x, u) = q(x)u \log(1 + |u|^s), \quad s > 0,$$

satisfy these assumptions if q is continuous and $q > 0$ on Ω, $q \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. Moreover, if f satisfies (f_0)–(f_4) and $g : \mathbb{R}^N \times \mathbb{R} \to [0, \infty)$ is continuous, vanishes outside of a bounded subset of $\Omega \times [0, \infty)$ and satisfies (f_2), then the sum $f + g$ also satisfies (f_0)–(f_4).

Our main result is the following:

Theorem 1.1. *Suppose that the nonlinearity f satisfies assumptions (f_0) – (f_4). Then the problem (4), (6) admits a solution in $H^1_{loc}(\mathbb{R}^N, \mathbb{R})$.*
If, in addition, $f(x, -t) = -f(x, t)$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$, then the problem (4), (6) admits a sequence of pairs of solutions $\{\pm u_n\}$, $n \in \mathbb{N}$ with the property that, for some $R > 0$,

$$\|u_n\|_{H^1(B_R)} \to \infty \qquad \text{as } n \to \infty. \tag{7}$$

Some remarks seem in order. First, we note that elliptic regularity theory implies that the solutions given by Theorem 1.1 belongs to $W^{2,q}_{loc}(\mathbb{R}^N)$ for all $1 \le q < \infty$, so there are strong solutions of (4) in $\mathcal{C}^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $0 < \alpha < 1$. Moreover, the restrictions of these solutions to $\mathbb{R}^N \setminus B_R$ are $\mathcal{C}^\infty$-functions, so the limit in (6) can be understood in strong sense.
In the proof of Theorem 1.1, we always work with a fixed value of R such that $\Omega \subset B_R$. This gives rise to the question whether or not the variational approach for different choices of R produces different solutions. This question is closely linked to the explicit representation of the capacity operator (and its real part) derived in Section 6 below, and we will give a partial answer in Remark 6.3. More precisely, if $R > 0$ is fixed, then for all up to countably many choices of $R' > R$ our approach yields different solutions when R is replaced by R'. Somewhat surprisingly at first glance, this fact gives rise to infinitely many solutions u_n, $n \in \mathbb{N}$ of (4), (6) also in the case where f is *not* odd in its second variable. However, a local unboundedness property as in (7) should not be expected for these solutions. We believe that this type of multiplicity has to be seen as a consequence of the fact that the asymptotic condition (6) is weaker than the Sommerfeld radiation condition (5).
It is natural to ask whether it is possible to relax the condition that f vanishes outside a compact set. We have no general answer to this question yet, but in the case where f is radial in x, a shooting argument yields radial solutions of (4), (6) under much less restrictive assumptions on f. For a precise result, see Theorem 5.2 below. We do not even need to assume that f tends to 0 as $r = |x| \to \infty$.

The paper is organized as follows. In Section 2, we will set up the variational framework used to prove Theorem 1.1. Here we also state key properties of the capacity operator T_R and its real part K_R in the case $N \ge 2$, but we postpone the derivation of these properties to Section 6 since the underlying computations — relying on special properties of Hankel functions — are somewhat technical. In Section 3 we then complete the proof of Theorem 1.1 in the case $N \ge 2$. As already remarked above, a key difficulty in the proof is the validity

of Cerami's condition (see Proposition 3.2), and the proof of this property is rather long. In Section 3, we also establish — under stronger assumptions on the nonlinearity — a rigid minimax principle with respect to families of half spaces for the solution which minimizes the corresponding energy functional among all critical points, see Theorem 3.5. We conjecture that this minimax principle gives rise to additional properties of the corresponding (ground state) solutions. This will be considered in future work by the authors. Section 4 contains a sketch of the proof of Theorem 1.1 in the one-dimensional case. This case is much easier than the case $N \geq 2$, but has to be treated slightly differently. Section 5 is devoted to the radial case. As pointed out already, we will apply a shooting argument to prove the existence of infinitely many radial solutions of (4), (6) under much less restrictive assumptions on f, see Theorem 5.2. Finally, as noted above, in Section 6 we derive key properties of the capacity operator T_R and its real part K_R. We note that some of these properties are well known (see e.g. [19] for the case $N = 3$), but we could not find an appropriate reference for general $N \geq 2$. Moreover, it seems that the operator K_R has not been studied in the degree of detail which we need for our purposes.

2. The variational framework

We assume that $N \geq 2$ in the next two sections, referring to Section 4 for the case $N = 1$. In this section we will introduce the capacity operator and develop a variational framework for problem (4), (6). We start by fixing some notation. Let $R > 0$ be such that $\Omega \subset B_R$, where $B_R := B_R(0)$ is the open ball with radius R centered at zero. We also set $E_R := \mathbb{R}^N \setminus \overline{B_R}$ and consider the space

$$\begin{aligned} H_R^{\mathbb{C}} := \Big\{ u \in H^1_{loc}(E_R,\mathbb{C}) \, : \frac{u}{(1+r^2)^{1/2}} \in L^2(E_R,\mathbb{C}),\ \frac{\nabla u}{(1+r^2)^{1/2}} \in L^2(E_R,\mathbb{C}^N), \\ \frac{\partial u}{\partial r} - iku \in L^2(E_R,\mathbb{C}) \Big\}, \end{aligned} \tag{8}$$

where — here and in the following — r always denotes the radial variable, i.e., $r = |x|$. It is known (see [19] for the case $N = 3$ and Section 6 below for general $N \geq 2$) that for every $u \in H^{\frac{1}{2}}(S_R,\mathbb{C})$, $S_R := \partial B_R$, there exists a unique weak solution $w \in H_R^{\mathbb{C}}$ of the problem

$$\begin{cases} \Delta w + k^2 w = 0 & \text{in } E_R, \\ w = u & \text{on } S_R. \end{cases} \tag{9}$$

Here weak solution means that $\text{trace}(w) = u \in H^{\frac{1}{2}}(S_R,\mathbb{C})$ and

$$\int_{E_R} \Big(\nabla w \cdot \nabla \varphi - k^2 w \varphi \Big)\, dx = 0 \qquad \text{for all } \varphi \in \mathcal{C}^1_c(E_R).$$

where $\mathcal{C}^1_c(E_R)$ denotes the space of $\mathcal{C}^1$-functions with compact support in E_R. We then define the *capacity operator* (or *Dirichlet to Neumann map*)

$$T_R : H^{\frac{1}{2}}(S_R,\mathbb{C}) \to H^{-\frac{1}{2}}(S_R,\mathbb{C}), \qquad T_R u = \frac{\partial w}{\partial \eta} \in H^{-\frac{1}{2}}(S_R,\mathbb{C}) \tag{10}$$

where $w \in H_R^{\mathbb{C}}$ is the unique solution of (9) corresponding to $u \in H^{\frac{1}{2}}(S_R,\mathbb{C})$ and η denotes the normal unit vector field on S_R pointing outside B_R and inside E_R.

As shown in [19] for the case $N = 3$ and detailed in Section 6 below for general $N \geq 2$, the operator $T_R : H^{\frac{1}{2}}(S_R,\mathbb{C}) \to H^{-\frac{1}{2}}(S_R,\mathbb{C})$ is continuous, and suitably normalized spherical harmonics (when considered as functions of spherical angles) form an orthonormal basis of

eigenfunctions of T_R. The operator T_R is not symmetric and therefore does not give rise to a variational formulation of our nonlinear problem given by (4) and (6). Therefore, we set $H^{\frac{1}{2}}(S_R) = H^{\frac{1}{2}}(S_R, \mathbb{R})$, $H^{-\frac{1}{2}}(S_R) = H^{-\frac{1}{2}}(S_R, \mathbb{R})$, and we let

$$K_R : H^{\frac{1}{2}}(S_R) \to H^{-\frac{1}{2}}(S_R), \qquad K_R u = \mathrm{Re}[T_R u] \tag{11}$$

denote the real part of the restriction of T_R to $H^{\frac{1}{2}}(S_R)$. This operator can be seen as the Dirichlet to Neumann map corresponding to the problem (9) with real data u on S_R and solutions w given as a real part of a function in $H_R^{\mathbb{C}}$. The explicit calculations of w in terms of u in Section 6 below immediately imply that w satisfies the asymptotic conditions (6). Moreover, the operator K_R has the following properties.

Lemma 2.1. *The operator K_R is bounded, symmetric and negative definite. More precisely, there are constants $\gamma_R > 0$ and $\Gamma_R > 0$ such that*

$$\|K_R u\|_{H^{-\frac{1}{2}}(S_R)} \leq \Gamma_R \|u\|_{H^{\frac{1}{2}}(S_R)} \quad \textit{and} \quad \int_{S_R} u K_R u \, d\sigma \leq -\gamma_R \int_{S_R} u^2 \, d\sigma \tag{12}$$

for all $u \in H^{\frac{1}{2}}(S_R)$. Moreover,

$$\int_{S_R} v K_R u \, d\sigma = \int_{S_R} u K_R v \, d\sigma \qquad \textit{for } u, v \in H^{\frac{1}{2}}(S_R). \tag{13}$$

We postpone the proof of this lemma to Section 6 below. Setting $X := H^1(B_R)$, we now recall the standard estimate

$$\int_{B_R} u^2 \, dx \leq c \int_{B_R} |\nabla u|^2 \, dx + c \int_{S_R} u^2 \, d\sigma \qquad \text{for } u \in X \tag{14}$$

with some constant $c = c(R) > 0$, see e.g. [25, Theorem A.9]. Moreover, we consider the bilinear form

$$\mathcal{B}_k : X \times X \to \mathbb{R}, \qquad \mathcal{B}_k(u, v) = \int_{B_R} \Big(\nabla u \cdot \nabla v - k^2 uv \Big) \, dx - \int_{S_R} v K_R u \, d\sigma. \tag{15}$$

From (12) and (14), we easily deduce the following

Corollary 2.2. *$\mathcal{B}_0$ defines a scalar product on $X = H^1(B_R)$ which is equivalent to the standard scalar product, i.e. the corresponding norms are equivalent.*

We also set $\mathcal{B}_k(u) := \mathcal{B}_k(u, u)$ for $u \in X$ in the following. In the next lemma, we collect key facts concerning $\mathcal{B}_k$ and the (nonlocal) eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } B_R, \\ \dfrac{\partial u}{\partial \eta} = K_R u & \text{on } S_R. \end{cases} \tag{16}$$

Lemma 2.3. *(i) The eigenvalue problem* (16) *admits an unbounded sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and a corresponding system of analytic eigenfunctions e_j, $j \in \mathbb{N}$ which is complete in X.*
(ii) There exists a scalar product $\langle \cdot, \cdot \rangle$ — equivalent to the standard scalar product on $X = H^1(B_R)$ — and an orthogonal splitting $X = X^- \oplus X^0 \oplus X^+$ such that

$$\mathcal{B}_k(u) = \|u^+\|^2 - \|u^-\|^2 \quad \textit{for all } u \in X,$$

where $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$ *is the induced norm, and* $u^\pm, u^0$ *denote the corresponding orthogonal projections of* u *onto* $X^\pm$, X^0, *respectively. More precisely,*

$$X^- = \mathrm{span}\{e_j\ : j \le j_*\}, X^0 = \mathrm{span}\{e_j\ : j_* < j < j^*\}\ \textit{and}\ X^+ = \overline{\mathrm{span}\{e_j\ : j \ge j^*\}},$$

where $j_* := \max\{j \in \mathbb{N}\ :\ \lambda_j < k^2\}$ *and* $j^* := \min\{j \in \mathbb{N}\ :\ \lambda_j > k^2\}$. *In particular,* X^- *and* X^0 *are finite dimensional.*
(iii) The family e_j, $j \in \mathbb{N}$ *is orthogonal with respect to the scalar products* $\langle\cdot,\cdot\rangle$, $\mathcal{B}_0$ *and the scalar product of* $L^2(B_R)$.
(iv) There exists a countable set $\mathcal{D} \subset (0,\infty)$ *such that* $X^0 \neq \{0\}$ *if and only if* $R \in \mathcal{D}$.

Proof. (i) We first consider $H^1(B_R)$ with the equivalent scalar product $\mathcal{B}_0$. Since K_R is negative definite, all eigenvalues of (16) must be positive. Hence $u \in H^1(B_R)$ is an eigenfunction of (16) corresponding to $\lambda \in \mathbb{R}$ if and only if $\lambda > 0$ and u is an eigenfunction of the operator $K \in \mathcal{L}(X)$ defined by

$$\mathcal{B}_0(Ku, v) = \int_{B_R} uv\,dx \qquad \text{for } u, v \in X \tag{17}$$

corresponding to the eigenvalue $\frac{1}{\lambda}$. The operator K is bounded, symmetric with respect to $\mathcal{B}_0$, nonnegative and compact, since the embedding $X \hookrightarrow L^2(B_R)$ is compact. Moreover, 0 is not an eigenvalue of K. Hence K admits a sequence of positive eigenvalues $\mu_1 \ge \mu_2 \ge \ldots$ such that $\mu_j \to 0$ as $j \to \infty$, and a corresponding complete system of $\mathcal{B}_0$-orthogonal eigenfunctions e_j, $j \in \mathbb{N}$. Assertion (i) now follows with $\lambda_j = \frac{1}{\mu_j}$, $j \in \mathbb{N}$ and the family $\{e_j\ :\ j \in \mathbb{N}\}$ thus obtained.
(ii) For $u \in X$, let $u^\pm, u^0$ denote the $L^2(B_R)$-orthogonal projections of u onto the subspaces $X^\pm$, X^0, respectively, as defined in the assertion. For $u, v \in X$, we define

$$\langle u, v\rangle = \mathcal{B}_k(u^+, v^+) - \mathcal{B}_k(u^-, v^-) + \int_{B_R} u^0 v^0\,dx.$$

It is easy to see that this scalar product has the desired properties, and by construction the splitting $X = X^- \oplus X^0 \oplus X^+$ is orthogonal with respect to this scalar product.
(iii) The $\mathcal{B}_0$-orthogonality of the family $\{e_j\ :\ j \in \mathbb{N}\}$ has already been shown above, and the $L^2(B_R)$-orthogonality then follows from (17). From this, the orthogonality with respect to $\langle\cdot,\cdot\rangle$ immediately follows by definition.
(iv) This part, which relies on special properties of the capacity operator and Hankel functions, will be proved in Section 6 below. □

We now consider the functional

$$\Phi : X \to \mathbb{R}, \qquad \Phi(u) = \frac{1}{2}\mathcal{B}_k(u) - \varphi(u) = \frac{1}{2}\Big(\|u^+\|^2 - \|u^-\|^2\Big) - \varphi(u), \tag{18}$$

where $\varphi(u) = \int_{B_R} F(x, u(x))\,dx$ for $u \in X$ and $F(x,t) = \int_0^t f(x,s)\,ds$ for $t \in \mathbb{R}$. It is well known that $\varphi \in \mathcal{C}^1(X, \mathbb{R})$ as a consequence of assumption (f_1), and

$$\varphi \text{ is nonnegative on } X \tag{19}$$

by (f_3). Moreover, the critical points of Φ correspond to restrictions to B_R of solutions of (4). Indeed, if $u \in X$ is a critical point of Φ, then

$$0 = \int_{B_R} \Big(\nabla u \cdot \nabla w - k^2 uw - f(x,u)w\Big)\,dx - \int_{S_R} wK_R u\,d\sigma$$

for every $w \in X$, hence u is a weak solution of the problem

$$\begin{cases} -\Delta u - k^2 u = f(x,u) & \text{in } B_R, \\ \dfrac{\partial u}{\partial \eta} = K_R u & \text{on } S_R. \end{cases} \tag{20}$$

As explained in Section 6, extending u by the real part of the unique solution of (9) in $H_R^{\mathbb{C}}$ then yields a solution of (4), (6).

3. Proof of Theorem 1.1

Using Lemma 2.3 (iii) and (iv), we will assume in the following that $R > 0$ is chosen such that $X^0 = \{0\}$. This restriction is only made in order to simplify the proofs, whereas all results can still be proved — with additional effort — in the case where X^0 has positive dimension.

We first collect useful facts about the functional Φ defined in (18). For this we recall the following well-known consequence of (f_1) and (f_2):

$$\forall\, \varepsilon > 0,\ \exists\, C_\varepsilon > 0 \text{ such that } |f(x,u)| \le \varepsilon|u| + C_\varepsilon |u|^{p-1} \text{ for all } (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{21}$$

Lemma 3.1. *(i) There exists $\alpha_0 > 0$ such that $\inf_{\Sigma_\alpha} \Phi > 0$ for $\alpha \in (0, \alpha_0)$, where $\Sigma_\alpha := \{u \in X^+ : \|u\| = \alpha\}$.*

(ii) Let $\mathcal{Z}$ be a closed cone contained in a finite dimensional subspace W of X and such that

$$\begin{aligned} &\{x \in \Omega \,:\, w(x) \neq 0\} \text{ has positive measure for} \\ &\text{every } w \in \mathcal{Z} \setminus \{0\} \text{ with } \|w^+\| \ge \|w^-\|. \end{aligned} \tag{22}$$

Then there exists $\rho = \rho(\mathcal{Z}) > 0$ such that $\Phi(u) \le 0$ for all $u \in \mathcal{Z}$ satisfying $\|u\| \ge \rho$.

Here and in the following, a set $\mathcal{Z} \subset X$ is called a cone if $\lambda x \in \mathcal{Z}$ for every $x \in \mathcal{Z}$, $\lambda \ge 0$. In particular, (ii) applies to $\mathcal{Z} = W$ if W is a finite dimensional subspace of X.

Proof. (i) For $u \in X^+$ we have $\Phi(u) = \frac{1}{2}\|u\|^2 - \varphi(u)$ and $\varphi(u) = o(\|u\|^2)$ as $u \to 0$ by (21) and Sobolev embeddings. Hence the conclusion follows.
(ii) Suppose by contradiction that a sequence $(u_n)_n \subset \mathcal{Z}$ exists with $\Phi(u_n) > 0$ for all n and $\|u_n\| \to \infty$ as $n \to \infty$. Setting $w_n = \frac{u_n}{\|u_n\|}$, we may pass to a subsequence such that $w_n \to w \in W$ since W is finite dimensional. Since $w_n \in \mathcal{Z}$ and $\|w_n\| = 1$ for all n, we have $w \in \mathcal{Z}$ and $\|w\| = 1$. Moreover, by (19) we have

$$0 \le \liminf_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^2} \le \frac{1}{2} \lim_{n\to\infty} (\|w_n^+\|^2 - \|w_n^-\|^2) = \frac{1}{2}(\|w^+\|^2 - \|w^-\|^2)$$

and therefore $\|w^+\| \ge \|w^-\|$. Hence (22) implies that $\Omega_w := \{x \in \Omega \,:\, w(x) \neq 0\}$ has positive measure. Passing to a subsequence, we may also assume that $w_n \to w$ pointwise a.e. in B_R, which implies that

$$|u_n(x)| = \|u_n\| |w_n(x)| \to \infty \qquad \text{as } n \to \infty \text{ for a.e. } x \in \Omega_w.$$

By (f_3) and Fatou's Lemma, we therefore deduce that

$$0 \le \frac{\Phi(u_n)}{\|u_n\|^2} \le \frac{1}{2}(\|w_n^+\|^2 - \|w_n^-\|^2) - \int_{\Omega_w} \frac{F(x,u_n)}{u_n^2} w_n^2 \, dx \to -\infty$$

as $n \to \infty$. This contradiction proves the claim. $\square$

To proceed with the proof of Theorem 1.1, we shall decompose the nonlinearity f as follows. We write $f = f_1 + f_2$, where $f_i : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$ are defined by

$$f_1(x,u) = \begin{cases} f(x,u), & |u| \geq s_0, \\ \dfrac{u^2}{s_0^2} f(x,s_0), & 0 \leq u \leq s_0, \\ \dfrac{u^2}{s_0^2} f(x,-s_0), & -s_0 \leq u \leq 0, \end{cases}$$

and we put $f_2 = f - f_1$. Setting $F_i(x,u) = \int_0^u f_i(x,s)\,ds$, $i = 1, 2$ for $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, we see that

$$\begin{aligned} &f_2(x,u) = 0 \text{ if } x \in \mathbb{R}^N \setminus \Omega \text{ or } |u| \geq s_0; \\ &F_2(x,u) = 0 \text{ if } x \in \mathbb{R}^N \setminus \Omega,\ u \in \mathbb{R}; \\ &f_2 \text{ and } F_2 \text{ are bounded on } \mathbb{R}^N \times \mathbb{R}. \end{aligned} \tag{23}$$

Moreover, f_1 satisfies condition (f_2) and the following stronger version of condition (f_4). For every $x \in \mathbb{R}^N$,

$$u \mapsto \frac{f_1(x,u)}{|u|} \text{ is nondecreasing on } \mathbb{R} \setminus \{0\}. \tag{24}$$

We decompose the functional $\varphi : X \to \mathbb{R}$ accordingly and write $\varphi = \varphi_1 + \varphi_2$ with

$$\varphi_i(u) = \int_{B_R} F_i(x,u(x))\,dx = \int_\Omega F_i(x,u(x))\,dx, \qquad i = 1, 2.$$

We note that φ_2 is bounded on X by (23). The following proposition will be a main technical step in the proof of Theorem 1.1.

Proposition 3.2. *Φ satisfies the Cerami condition in X, i.e., every sequence $(u_n)_n \subset X$ such that $\Phi(u_n) \to c$ for some $c \in \mathbb{R}$ and $(1 + \|u_n\|)\|\Phi'(u_n)\| \to 0$ as $n \to \infty$ has a subsequence which converges in X.*

The proof of this proposition is quite long and requires subtle estimates. Parts of the proof are inspired by [17] and [16], but we need new arguments to deal with the difficulty that the nonlinearity may vanish on a subset of B_R of positive measure. A key role in the proof is played by the useful inequality

$$\begin{aligned} &f_1(x,u)[s(\frac{s}{2}+1)u + (1+s)v] + F_1(x,u) - F_1(x,[1+s]u+v) \leq 0 \\ &\text{for } x \in \mathbb{R}^N,\ u, v \in \mathbb{R} \text{ and } s \geq -1, \end{aligned} \tag{25}$$

which follows from (24). Indeed, as noted in [17], this inequality is a weak version of [27, Lemma 2.2]. As a consequence of (25) and properties of f_2, we may derive the following

Lemma 3.3. *For every $K > 0$ there is a constant $C = C(K) > 0$ with the following property. If $u, s, v \in \mathbb{R}$ are numbers with $-1 \leq s \leq K$ and $|v| \leq K$, then*

$$f(x,u)[s(\frac{s}{2}+1)u + (1+s)v] + F(x,u) - F(x,[1+s]u+v) \leq C \qquad \text{for all } x \in \mathbb{R}^N.$$

Proof. By (23) there exists a constant $C_1 > 0$ (depending on K) such that

$$|f_2(x,u)| \leq C_1, \quad |F_2(x,u)| \leq C_1 \quad \text{and} \quad |f_2(x,u)s(\frac{s}{2}+1)u| \leq C_1$$

for $u \in \mathbb{R}$, $x \in \mathbb{R}^N$, $|s| \le K+1$. Consequently, we have

$$\begin{aligned} f_2(x,u)[s(\frac{s}{2}+1)u+(1+s)v] + F_2(x,u) - F_2(x,[1+s]u+v) \\ \le C_1[1+K(K+1)] + 2C_1 \end{aligned}$$

for $x \in \mathbb{R}^N$ and $u, s, v \in \mathbb{R}$ with $-1 \le s \le K$ and $|v| \le K$. Since $f = f_1 + f_2$, $F = F_1 + F_2$ and (25) holds, the claim follows with $C := C_1[1+K(K+1)] + 2C_1$. □

The next step in the proof of Proposition 3.2 is the following relative energy estimate. In the following, a sequence $(u_n)_n$ in X is called a *Cerami sequence* for Φ if $\Phi(u_n) \to c$ for some $c \in \mathbb{R}$ and $(1+\|u_n\|)\|\Phi'(u_n)\| \to 0$ as $n \to \infty$.

Lemma 3.4. *For every $\kappa > 0$ there exists $\tilde{C} = \tilde{C}(\kappa) > 0$ with the following property. If $(u_n)_n$ is a Cerami sequence for Φ, and $r_n \ge 0$, $v_n \in X^-$, $n \in \mathbb{N}$ are given with $r_n \le \kappa$ and $\|v_n\| \le \kappa$ for all $n \in \mathbb{N}$, then*

$$\Phi(r_n u_n + v_n) \le \Phi(u_n) + \tilde{C} + o(1) \qquad \textit{as } n \to \infty.$$

Proof. We first note that, by a standard bootstrap argument using elliptic regularity theory, there exists a constant $K = K(\kappa) > \kappa$ such that

$$\|v\|_{L^\infty(B_R)} \le K \qquad \text{for every } v \in X^- \text{ with } \|v\| \le \kappa.$$

We write $r_n = 1 + s_n$ with $-1 \le s_n \le \kappa - 1 \le K$ and set $w_n = r_n u_n + v_n = (1+s_n)u_n + v_n$ for $n \in \mathbb{N}$. Then

$$\begin{aligned} \Phi(w_n) - \Phi(u_n) &= \frac{1}{2}[\mathcal{B}_k(w_n) - \mathcal{B}_k(u_n)] + \int_{B_R} (F(x,u_n) - F(x,w_n))\,dx \\ &= \frac{1}{2}\Big([(1+s_n)^2 - 1]\mathcal{B}_k(u_n) + 2(1+s_n)\mathcal{B}_k(u_n,v_n) + \mathcal{B}_k(v_n)\Big) \\ &\qquad + \int_{B_R} (F(x,u_n) - F(x,w_n))\,dx \\ &= -\frac{\|v_n\|^2}{2} + \mathcal{B}_k(u_n, s_n(\frac{s_n}{2}+1)u_n + (1+s_n)v_n) + \int_{B_R} (F(x,u_n) - F(x,w_n))\,dx \\ &\le \mathcal{B}_k(u_n, s_n(\frac{s_n}{2}+1)u_n + (1+s_n)v_n) + \int_{B_R} (F(x,u_n) - F(x,w_n))\,dx \end{aligned}$$

Since $(u_n)_n$ is a Cerami sequence, $\|v_n\| \le \kappa$ and $|s_n| \le K+1$ for all n, we have

$$\begin{aligned} &\Big|\mathcal{B}_k(u_n, s_n(\frac{s_n}{2}+1)u_n + (1+s_n)v_n) - \int_{\mathbb{R}^N} f(x,u_n)[s_n(\frac{s_n}{2}+1)u_n + (1+s_n)v_n]\,dx\Big| \\ &= \Big|\Phi'(u_n)\Big(s_n(\frac{s_n}{2}+1)u_n + (1+s_n)v_n\Big)\Big| \\ &\le c_1\|\Phi'(u_n)\|\|u_n\| + c_2\|\Phi'(u_n)\|\|v_n\| = o(1) \end{aligned}$$

as $n \to \infty$ with constants $c_1, c_2 > 0$ (depending on K). Consequently,

$$\begin{aligned} \Phi(w_n) - \Phi(u_n) \le \int_{B_R} \Big(f(x,u_n)[s_n(\frac{s_n}{2}+1)u_n + (1+s_n)v_n] \\ + F(x,u_n) - F(x,w_n)\Big)\,dx + o(1). \end{aligned}$$

Choosing $C = C(K)$ as in Lemma 3.3, we conclude that

$$\Phi(w_n) - \Phi(u_n) \le |B_R|C + o(1) \qquad \text{as } n \to \infty.$$

Hence the assertion follows with $\tilde{C} = |B_R|C + 1$. □

We may now complete the

Proof of Proposition 3.2. Let $(u_n)_n$ be a sequence with the assumed properties. We first show that $(u_n)_n$ is bounded in X. Assuming by contradiction that this is false, we may pass to a subsequence — still denoted by $(u_n)_n$ — such that $\|u_n\| \to \infty$ as $n \to \infty$. Setting $w_n := \frac{u_n}{\|u_n\|}$, we may assume, passing again to a subsequence, that $w_n \rightharpoonup w$ weakly in X for some $w \in X$, $w_n \to w$ in $L^q(B_R)$ for all $1 \le q < 2^*$, and $w_n \to w$ pointwise a.e. on B_R as $n \to \infty$. Moreover, we have $w_n^+ \rightharpoonup w^+$ weakly in X and $w_n^- \to w^-$ strongly in X as $n \to \infty$, since X^- is finite-dimensional. Passing to a further subsequence, we may also assume that $\|w_n^+\| \to s \ge 0$ as $n \to \infty$ for some $s \ge \|w^+\|$. Since

$$o(1) = \frac{\Phi(u_n)}{\|u_n\|^2} \le \frac{1}{2}\Big(\|w_n^+\|^2 - \|w_n^-\|^2\Big),$$

by (19), we find that

$$\|w^-\| \le s. \tag{26}$$

Hence, $1 = \|w_n\|^2 = \|w_n^+\|^2 + \|w_n^-\|^2 \to s^2 + \|w^-\|^2 \le 2s^2$ as $n \to \infty$, indicating that $s > 0$. Next we suppose by contradiction that

$$w \equiv 0 \qquad \text{a.e. on } \Omega \tag{27}$$

which implies that

$$\varphi(tw_n) \to 0 \qquad \text{as } n \to \infty \text{ for all } t > 0. \tag{28}$$

We claim that there exist $t > 0$ and $v_n \in X^-$, $n \in \mathbb{N}$ with

$$\|v_n\| \le 1 \quad \text{and} \quad \Phi(tw_n + v_n) > c + \tilde{C} + 1 \qquad \text{for } n \text{ sufficiently large,} \tag{29}$$

where $\tilde{C} = \tilde{C}(1)$ is chosen as in Lemma 3.4 corresponding to $\kappa = 1$. To prove this, we have to distinguish different cases. We first note that

$$\Phi(tw_n) = \frac{t^2}{2}\Big(s^2 - \|w^-\|^2\Big) + o(1) \quad \text{as } n \to \infty$$

for every $t > 0$ by (28). Hence, if $\|w^-\| < s$, we can find $t > 0$ such that

$$\Phi(tw_n) > c + \tilde{C} + 1 \qquad \text{for } n \text{ sufficiently large,}$$

and therefore (29) follows with $v_n = 0$ for every n.
Next we consider the remaining case $\|w^-\| = s$, and we note that for every $t > 0$ we have $tw_n + w_n^+ \to tw + w^+$ in $L^q(B_R)$ for all $1 \le q < 2^*$ and also pointwise a.e. on B_R. Hence

$$\varphi(tw_n + w_n^+) \to \varphi(tw + w^+) = \varphi(w^+) \qquad \text{as } n \to \infty \tag{30}$$

for all $t > 0$ by (27). Consequently

$$\Phi(tw_n + w_n^+) = \frac{1}{2}[s^2(t+1)^2 - t^2 s^2] - \varphi(w^+) + o(1) = s^2(t + \frac{1}{2}) - \varphi(w^+) + o(1),$$

so that there exists $t > 0$ such that

$$\Phi(tw_n + w_n^+) > c + \tilde{C} + 1 \qquad \text{for } n \text{ sufficiently large.}$$

Again, (29) follows with $t+1$ in place of t and $v_n = -w_n^-$, since $tw_n + w_n^+ = (t+1)w_n - w_n^-$ for every n. Next, fixing $t > 0$ and v_n, $n \in \mathbb{N}$ such that (29) holds, we write $tw_n + v_n = s_n u_n + v_n$

with $s_n = \frac{t}{\|u_n\|}$ for every n, so that $0 < s_n \le 1$ for large n and $\|v_n\| \le \|w_n\| = 1$. By Lemma 3.4, we therefore have

$$\Phi(tw_n + v_n) = \Phi(s_n u_n + v_n) \le \Phi(u_n) + \tilde{C} + o(1)$$

as $n \to \infty$, which contradicts (29). The contradiction shows that (27) is false, and therefore the set $\Omega_w := \{x \in \Omega : w(x) \neq 0\}$ has positive measure. Moreover,

$$|u_n(x)| = \|u_n\|\,|w_n(x)| \to +\infty \text{ as } n \to \infty \text{ for almost every } x \in \Omega_w.$$

Hence, Fatou's Lemma, the L^2-convergence $w_n \to w$ in B_R and (f_3) imply

$$o(1) = \frac{\Phi(u_n)}{\|u_n\|^2} \le \frac{1}{2} - \int_{\Omega_w} \frac{F(x,u_n)}{|u_n|^2}\,|w_n|^2\,dx \to -\infty,$$

as $n \to \infty$, a contradiction. The contradiction shows that $(u_n)_n$ is bounded in X. Therefore, we can find a subsequence — still denoted by $(u_n)_n$ — and some $u \in X$ such that $u_n \rightharpoonup u$ (weakly) in X, $u_n \to u$ in $L^q(B_R)$ for all $1 \le q < 2^*$ and $u_n(x) \to u(x)$ for a.e. $x \in B_R$. As a consequence of (21), there holds

$$\int_{B_R} (f(x,u_n) - f(x,u))(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\begin{aligned}\mathcal{B}_0(u_n - u, u_n - u) = (\Phi'(u_n) - \Phi'(u))(u_n - u) + k^2 \int_{B_R} (u_n - u)^2\,dx \\ + \int_{B_R} (f(x,u_n) - f(x,u))(u_n - u)\,dx \to 0,\end{aligned}$$

as $n \to \infty$. It follows from Corollary 2.2 that $u_n \to u$ strongly in X. The proof is finished. □

We may now complete the

Proof of Theorem 1.1 (Case $N \ge 2$). The existence of a nontrivial solution follows from a variant of the classical linking theorem where the Palais-Smale condition is replaced by the Cerami condition [4, Theorem 2.3]. To see this, we proceed as follows. Considering the sequence of eigenfunctions $(e_j)_{j\in\mathbb{N}}$ of (16) given by Lemma 2.3, we set $u = e_{j^*} \in X^+$ and put

$$Q_\rho := \{tu + v \,:\, v \in X^-,\ \|v\| \le \rho,\ 0 \le t \le \rho\} \qquad \text{for } \rho > 0.$$

Note that the sets Q_ρ are contained in the finite dimensional subspace $W = X^- \oplus \mathbb{R}u \subset X$. Since every function in W is analytic in B_R, $\mathcal{Z} := W$ satisfies condition (22). Using Lemma 3.1(ii) and the fact that Φ is nonpositive on X^- by (19), we thus find that

$$\sup_{\partial Q_\rho} \Phi = 0 \qquad \text{for } \rho > 0 \text{ sufficiently large.}$$

According to Lemma 3.1(i), we may further choose $\alpha > 0$ sufficiently small such that the sets Σ_α and ∂Q_ρ link and $\inf_{\Sigma_\alpha} \Phi > 0$ (see e.g. [25] or [4, Section 2] for the notion of linking of sets). Finally, we have $\sup_{Q_\rho} \Phi < +\infty$ by the compactness of Q_ρ. Taking Proposition 3.2 into account, we can apply the linking theorem and obtain that Φ has a nontrivial critical point $\hat{u} \in X$ such that

$$0 < \inf_{\Sigma_\alpha} \Phi \le \Phi(\hat{u}) \le \sup_{Q_\rho} \Phi.$$

Let us now assume that $f(x,-t) = -f(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. In this case, a variant of the Fountain Theorem of Bartsch (see [5, 7] and [30, Theorem 3.6]) yields the existence of a sequence of pairs $\{\pm u_n\}$, $n \in \mathbb{N}$ of critical points of Φ such that

$$\Phi(u_n) \to \infty \qquad \text{as } n \to \infty. \tag{31}$$

More precisely, we use a version of the Fountain Theorem where the Cerami condition is used instead of the Palais-Smale condition. To see that such a variant exists, it suffices to note the validity of a deformation lemma giving rise to Cerami sequences instead of Palais-Smale sequences. Such a deformation lemma has already been established in [4, Theorem 1.3]. In order to check the other assumptions of the Fountain Theorem, we remark that $X = \overline{\bigoplus_{j\in\mathbb{N}} \mathbb{R}e_j}$ where $(e_j)_{j\in\mathbb{N}}$ is given by Lemma 2.3(i). We set

$$X_j = \mathbb{R}e_j, \quad Y_j = \bigoplus_{\ell=1}^{j} X_\ell \quad \text{and} \quad Z_j = \overline{\bigoplus_{\ell=j}^{\infty} X_\ell}$$

for $j \in \mathbb{N}$. Since every function in Y_j is analytic, we see from Lemma 3.1(ii) — applied to $\mathcal{Z} = Y_j$ — that for every $j \in \mathbb{N}$ there exists $\rho_j > 0$ such that $\Phi(u) \le 0$ for $u \in Y_j$ with $\|u\| \ge \rho_j$. It only remains to check that for some sequence $(r_j)_j \subset (0,\infty)$

$$\inf\{\Phi(u) \,:\, u \in Z_j,\ \|u\| = r_j\} \to \infty \text{ as } j \to \infty. \tag{32}$$

This will be proved by similar arguments as in [30, Theorem 3.7]. Indeed, if $j \ge j^*$, then $Z_j \subset X^+$ and therefore, by (21),

$$\begin{aligned}\Phi(u) = \frac{1}{2}\|u\|^2 - \int_{B_R} F(x,u)\,dx &\ge \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\|u\|^2_{L^2(B_R)} - \frac{C_\varepsilon}{p}\|u\|^p_{L^p(B_R)} \\ &\ge \frac{1}{4}\|u\|^2 - \frac{C_\varepsilon}{p}\beta_j^p\|u\|^p \qquad \text{for } u \in Z_j,\end{aligned} \tag{33}$$

where

$$\varepsilon = \frac{1}{2}\inf_{u\in X\setminus\{0\}} \frac{\|u\|^2}{\|u\|^2_{L^2(B_R)}} > 0 \quad \text{and} \quad \beta_j := \sup\{\|u\|_{L^p(B_R)} \,:\, u \in Z_j,\ \|u\| = 1\}.$$

Since $(\beta_j)_j \subset [0,\infty)$ is a decreasing sequence, $\beta := \lim_{j\to\infty} \beta_j$ exists. Moreover, for each j we can find some $u_j \in Z_j$ such that $\|u_j\| = 1$ and $\beta_j \ge \|u_j\|_{L^p(B_R)} > \frac{\beta_j}{2}$. From the definition of Z_j we obtain that $u_j \rightharpoonup 0$ (weakly) in X and the compact Sobolev embedding $X \hookrightarrow L^p(B_R)$ then gives $u_j \to 0$ in $L^p(B_R)$. Thus, $\beta_j \to 0$ as $j \to \infty$. Choosing $r_j = (2C_\varepsilon\beta_j^p)^{\frac{1}{2-p}}$, we obtain from (33) that

$$\Phi(u) \ge r_j^2\Big(\frac{1}{4} - \frac{1}{2p}\Big) \qquad \text{for all } u \in Z_j \text{ with } \|u\| = r_j.$$

Since $r_j \to \infty$ as $j \to \infty$, the assertion follows. Moreover, since Φ is bounded on bounded subsets of $X = H^1(B_R)$, (31) implies that $\|u_n\|_{H^1(B_R)} \to \infty$ as $n \to \infty$, as claimed in (7).
To conclude the proof of Theorem 1.1, we remark that if u is a critical point of Φ, then the restriction of u to S_R belongs to $H^{\frac{1}{2}}(S_R)$ and there exists a unique weak solution w of (9). Therefore, extending u on $\mathbb{R}^N$ by setting $u \equiv \mathrm{Re}(w)$ on E_R, we see that $u \in H^1_{loc}(\mathbb{R}^N)$ is a weak solution of (4). Furthermore, since $u|_{E_R} = \frac{1}{2}(w + \overline{w})$ and w satisfies (5), we find that u satisfies the condition (6). $\square$

We close this section with an observation on a rigid minimax characterization of the ground state energy level (i.e., the least energy of a nontrivial critical point of Φ) in the case where (f_4) is replaced by a stronger condition.

Theorem 3.5. *Suppose that the nonlinearity f satisfies $(f_1)-(f_3)$ and the following stronger version of (f_4): For every $x \in \mathbb{R}^N$,*

$$u \mapsto \frac{f(x,u)}{|u|} \text{ is nondecreasing on } \mathbb{R} \setminus \{0\}. \tag{34}$$

Then the ground state energy level

$$c := \inf\{\Phi(u) \,:\, u \text{ nontrivial critical point of } \Phi\}$$

is positive and equivalently given by

$$c = \inf_{u \in X^+ \setminus \{0\}} \sup_{t \geq 0, v \in X^-} \Phi(tu + v)$$

Moreover, c is attained within the set of nontrivial critical points of Φ, i.e., within the set of nontrivial solutions of (20).

Proof. Let u be a nontrivial critical point of Φ. Then $u \notin X^-$, since otherwise

$$\int_{B_R} f(x,u)u\,dx = \mathcal{B}_k(u) = -\|u^-\|^2 < 0$$

contrary to (34) and (f_2). Next we claim that

$$\Phi(u) \geq \Phi(w) \qquad \text{for every } w \in X^- + \mathbb{R}^+ u \;=\; X^- + \mathbb{R}^+ u^+. \tag{35}$$

Indeed, similarly as in the proof of Lemma 3.4 we have, for $w = (1+s)u + v$ with $s \geq -1$ and $v \in X^-$,

$$\begin{aligned}\Phi(w) - \Phi(u) &\leq \mathcal{B}_k(u, s(\frac{s}{2}+1)u + (1+s)v) + \int_{B_R} (F(x,u) - F(x,w))\,dx \\ &= \int_{B_R} \Big(f(x,u)[s(\frac{s}{2}+1)u + (1+s)v] + F(x,u) - F(x,w) \Big)\,dx.\end{aligned}$$

Moreover, the last integral is nonpositive since, as a consequence of (34), the inequality (25) holds for f in place of f_1. Hence (35) is true. Putting

$$c^* = \inf_{u \in X^+ \setminus \{0\}} \sup_{t \geq 0, v \in X^-} \Phi(tu + v),$$

we thus infer that $c \geq c^*$, whereas Lemma 3.1(i) implies that $c^* \geq \inf_{\Sigma_\alpha} \Phi > 0$ for some $\alpha > 0$, where $\Sigma_\alpha := \{u \in X^+ \,:\, \|u\| = \alpha\}$.
Furthermore, c is attained among nontrivial critical points of Φ. Indeed, if $(u_n)_n$ is a sequence of critical points of Φ with $\Phi(u_n) \to c$, then by Proposition 3.2 we have $u_n \to u_0$ in X for a subsequence, where u_0 is a critical point of Φ with $\Phi(u_0) = c > 0$, and therefore $u_0 \neq 0$.
It remains to show that $c \leq c^*$. For this, let $u \in X^+ \setminus \{0\}$ be such that

$$\sup_{t \geq 0, v \in X^-} \Phi(tu + v) < \infty \tag{36}$$

Let $\mathcal{Z} := \{tu + v \;:\; v \in X^-,\ t \geq 0\}$. We claim that condition (22) holds for this set $\mathcal{Z}$. Suppose by contradiction that this is false, i.e., there exists $w \in \mathcal{Z} \setminus \{0\}$ with $\|w^+\| \geq \|w^-\|$

and such that $w \equiv 0$ a.e. on Ω. Since $\|w^+\| > 0$, we then find

$$\begin{aligned}\Phi(tw + w^+) &= \frac{1}{2}\Big((t+1)^2\|w^+\|^2 - t^2\|w^-\|^2\Big) - \varphi(w^+)\\ &\geq (t + \frac{1}{2})\|w^+\|^2 - \varphi(w^+) \to \infty\end{aligned}$$

as $t \to \infty$. This contradicts (36), since $tw + w^+ = (t+1)w^+ + tw^- \in \mathcal{Z}$ for every $t > 0$. The contradiction shows that condition (22) holds for $\mathcal{Z}$. Since $\mathcal{Z}$ is a closed cone contained in the finite dimensional space $X^- \oplus \mathbb{R}u$, Lemma 3.1(ii) implies that there exists $\rho > 0$ with

$$\sup_{\partial Q_\rho} \Phi = 0 \qquad \text{for} \quad Q_\rho := \{tu + v \,:\, v \in X^-,\ \|v\| \leq \rho,\ 0 \leq t \leq \rho\}. \tag{37}$$

Applying the linking theorem [4, Theorem 2.3] exactly as in the proof of Theorem 1.1 with the present choice of Q_ρ, we obtain a nontrivial critical point $\hat{u}$ of Φ with

$$\Phi(\hat{u}) \leq \max_{Q_\rho} \Phi \leq \sup_{t \geq 0, v \in X^-} \Phi(tu + v).$$

We thus conclude that $c \leq c^*$, so equality holds and the proof is finished. □

4. The one-dimensional case

In this section we sketch the proof of Theorem 1.1 in the case $N = 1$ which has to be treated slightly differently. Note that for $R > 0$ we have $B_R = (-R, R)$, $S_R := \{-R, R\}$ and $E_R = (-\infty, -R) \cup (R, +\infty)$, and consider the space $H_R^{\mathbb{C}}$ as defined in (8). For a given function $u : S_R \to \mathbb{C}$, it is easy to see that the unique solution $w \in H_R^{\mathbb{C}}$ of the problem

$$\begin{cases} w'' + k^2 w = 0 & \text{in } E_R, \\ w = u & \text{on } S_R \end{cases} \tag{38}$$

is then given by

$$w(x) = \begin{cases} u(-R)e^{-ik(x+R)} & \text{for } x \leq -R, \\ u(R)e^{ik(x-R)} & \text{for } x \geq R. \end{cases}$$

Hence the capacity operator T_R is simply given by $[T_R u](R) = iku(R)$ and $[T_R u](-R) = iku(-R)$ for any function $u : S_R \to \mathbb{C}$, and thus its real part K_R, as defined in (11), is zero. In particular, the second inequality in (12) is not true if $N = 1$, and also Corollary 2.2 does not hold in this case. On the other hand, the quadratic form $\mathcal{B}_k$ defined in (15) is simply given by

$$\mathcal{B}_k(u, v) = \int_{-R}^{R} (u'v' - k^2 uv)\, dx, \qquad u, v \in H^1(-R, R),$$

and (16) reduces to the homogeneous Neumann eigenvalue problem

$$-u'' = \lambda u \quad \text{in } (-R, R), \qquad u'(\pm R) = 0. \tag{39}$$

As a consequence, the properties listed in Lemma 2.3 are well known and easy to check in the case $N = 1$. Furthermore, the inner problem (20) reduces to

$$\begin{cases} -u'' - k^2 u = f(x, u) & \text{in } (-R, R), \\ u'(\pm R) = 0, \end{cases} \tag{40}$$

and solutions of (40) correspond to critical points of the functional

$$\Phi : H^1(-R,R) \to \mathbb{R}, \qquad \Phi(u) = \frac{1}{2}\int_{-R}^{R}\Big((u')^2 - k^2u^2\Big)\,dx - \int_{-R}^{R} F(x,u)\,dx.$$

Since the structure of Φ is the same as in the multidimensional case, the proof of Theorem 1.1 can be finished exactly as detailed in Section 3. In particular, the one-dimensional analogues of Lemma 3.1 and Proposition 3.2 are valid.

In addition, we remark that, by exactly the same proof, Theorem 3.5 is also true in the one-dimensional case.

5. Existence of radial solutions

For $N \geq 2$, we look for radially symmetric solutions of

$$-\Delta u - k^2 u = f(|x|, u), \qquad x \in \mathbb{R}^N, \tag{41}$$

where $k > 0$, and f satisfies the following assumptions

(g_1) $f \in \mathcal{C}([0,\infty)\times\mathbb{R})$ with $f(\cdot,0) = 0$, and f is locally Lipschitz continuous with respect to $u \in \mathbb{R}$.

(g_2) $0 \leq F(r,u) \leq \frac{1}{2}f(r,u)u$ for all $(r,u) \in [0,\infty)\times\mathbb{R}$, where $F(r,u) = \int\limits_0^u f(r,s)\,ds$.

(g_3) For every $u \in \mathbb{R}$, $F(\cdot,u) \in \mathcal{C}^1(0,\infty)$ with $\partial_r F(r,u) \leq 0$ for all $r > 0$.

Setting $u(x) = u(r)$ with $r = |x|$, we can rewrite (41) as

$$-u'' - \frac{N-1}{r}u' - k^2u = f(r,u), \quad r > 0 \tag{42}$$

and we shall look for solutions of class C^2 of this equation which satisfy $u'(0) = 0$.

The following local existence and uniqueness result (in a singular setting) is well known, see e.g. [29, Lemma 3.1] for a proof.

Lemma 5.1. *For every $\alpha \in \mathbb{R}$, there exists $\varepsilon > 0$ such that the initial value problem associated to* (42)*, $u(0) = \alpha$ and $u'(0) = 0$ has a unique solution $u \in \mathcal{C}^2([0,\varepsilon),\mathbb{R})$.*

Using properties (g_2) and (g_3), we now continue the local solution given by Lemma 5.1 to a global solution on $\mathbb{R}^+$ and analyze its asymptotic behavior.

Theorem 5.2. *Let f satisfy (g_1) to (g_3). Then for every $\alpha \in \mathbb{R}$, there exists a unique radially symmetric solution of* (41)*, $u = u(|x|)$, $u \in \mathcal{C}^2([0,\infty),\mathbb{R})$ satisfying $u(0) = \alpha$ and $u'(0) = 0$. Moreover,*

$$\sup_{r\geq 1}\big\{r^{N-1}\big(u'(r)^2 + u(r)^2\big)\big\} < \infty, \tag{43}$$

so that — identifying $u(x)$ with $u(|x|)$ for $x \in \mathbb{R}^N$ — we have $\frac{u}{\sqrt{1+|x|^2}}, \frac{|\nabla u|}{\sqrt{1+|x|^2}} \in L^2(\mathbb{R}^N)$. If, in addition,

(g_4) *$f(r,u) = o(|u|)$ as $u \to 0$, uniformly in $r > 0$,*

then u satisfies the asymptotic condition (6)*.*

Proof. Let $\alpha > 0$. By Lemma 5.1, there exists $\varepsilon > 0$ and a unique solution $u \in \mathcal{C}^2([0,2\varepsilon),\mathbb{R})$ satisfying (42) and $u(0) = \alpha$, $u'(0) = 0$. Let $[0,r_0)$, $r_0 \in [2\varepsilon,\infty]$ denote the maximal existence interval of this solution. We claim that

$$\sup_{r\in[\varepsilon,r_0)} r^{N-1}\big(u'(r)^2 + u(r)^2\big) < \infty, \tag{44}$$

To see this, we use the change of variables

$$v(r) = r^{\frac{N-1}{2}} u(r), \quad r \in (0, r_0),$$

so that v solves

$$-v'' - \{k^2 - \frac{(N-1)(N-3)}{4r^2}\}v = r^{\frac{N-1}{2}} f(r, r^{-\frac{N-1}{2}} v), \quad r \in (0, r_0). \tag{45}$$

We consider the $\mathcal{C}^1$-function $r \mapsto \rho(r) := v'(r)^2 + k^2 v(r)^2$ on $(0, r_0)$ which satisfies

$$\frac{1}{2}\rho'(r) = \frac{(N-1)(N-3)}{4r^2} v(r)v'(r) - r^{\frac{N-1}{2}} f(r, r^{-\frac{N-1}{2}} v(r))v'(r).$$

For $r \in [\varepsilon, r_0)$, we thus obtain

$$\begin{aligned}
\rho(r) &= \rho(\varepsilon) + 2\int_\varepsilon^r \frac{(N-1)(N-3)}{4s^2} v(s)v'(s)\, ds - 2\int_\varepsilon^r s^{\frac{N-1}{2}} f(s, s^{-\frac{N-1}{2}} v(s))v'(s)\, ds \\
&= \rho(\varepsilon) + 2\int_\varepsilon^r \frac{(N-1)(N-3)}{4s^2} v(s)v'(s)\, ds - 2s^{N-1} F(s, s^{-\frac{N-1}{2}} v(s))\Big|_\varepsilon^r \\
&\quad - (N-1)\int_\varepsilon^r s^{N-2}\{f(s, s^{-\frac{N-1}{2}} v(s))s^{-\frac{N-1}{2}} v(s) - 2F(s, s^{-\frac{N-1}{2}} v(s))\}\, ds \\
&\quad + 2\int_\varepsilon^r s^{N-1} \partial_r F(s, s^{-\frac{N-1}{2}} v(s))\, ds \\
&\le \rho(\varepsilon) + 2\varepsilon^{N-1} F(\varepsilon, \varepsilon^{\frac{1-N}{2}} v(\varepsilon)) + 2\int_\varepsilon^r \frac{(N-1)(N-3)}{4s^2} v(s)v'(s)\, ds \\
&\le \rho(\varepsilon) + 2\varepsilon^{N-1} F(\varepsilon, \varepsilon^{\frac{1-N}{2}} v(\varepsilon)) + \int_\varepsilon^r \frac{(N-1)|N-3|}{4ks^2} \rho(s)\, ds,
\end{aligned}$$

using (g_2) and (g_3). Hence, Gronwall's inequality (see [12, Theorem III.1.1]) gives

$$\rho(r) \le [\rho(\varepsilon) + 2\varepsilon^{N-1} F(\varepsilon, \varepsilon^{\frac{1-N}{2}} v(\varepsilon))]e^{\frac{(N-1)|N-3|}{4k\varepsilon^2}} \quad \text{for } r \in [\varepsilon, r_0).$$

From this we derive (44), since

$$r^{N-1}\big(u'(r)^2 + u(r)^2\big) \le C_\varepsilon \rho(r) \quad \text{for } r \in [\varepsilon, r_0) \text{ with some constant } C_\varepsilon > 0.$$

As a consequence of (44), we find that $r_0 = \infty$ and that (43) holds.
We finally assume that (g_4) holds, and we check that the radiation condition in (6) is fulfilled. For this we let $r \ge 1$ and write

$$\begin{aligned}
r^{\frac{N-1}{2}} \left|u''(r) + k^2 u(r)\right| &= \left| \frac{(N-1)}{r}\left(\frac{(N-1)}{2r} v(r) - v'(r)\right) - \frac{f(r, r^{-\frac{(N-1)}{2}} v(r))}{r^{-\frac{(N-1)}{2}} v(r)}\, v(r) \right| \\
&\le \frac{(N-1)^2}{2r^2}\|v\|_\infty + \frac{(N-1)}{r}\|v'\|_\infty + \left| \frac{f(r, r^{-\frac{(N-1)}{2}} v(r))}{r^{-\frac{(N-1)}{2}} v(r)} \right| \|v\|_\infty.
\end{aligned}$$

Since $\lim_{r\to\infty} r^{-\frac{(N-1)}{2}} v(r) = 0$, the assumption (g_4) gives $\lim_{r\to\infty} r^{\frac{N-1}{2}} \left|u''(r) + k^2 u(r)\right| = 0$, i.e., condition (6) holds. □

We point out that if f satisfies $(g_1) - (g_4)$, every solution of (41) given by Theorem 5.2 is oscillatory. This follows from the fact that for every such solution u, 0 lies in the essential spectrum of any self-adjoint realization of the Sturm-Liouville differential expression

$$\tau w = -w'' - \Big(k^2 - \frac{(N-1)(N-3)}{r^2} + \frac{f(r, u(r))}{u(r)}\Big)w$$

associated to (45) (see [10, Corollary XIII.7.14 and Theorem XIII.7.40]).

6. Properties of the capacity operator and its real part

In this section we derive key properties of the operator T_R introduced in Section 2 and its real part K_R in the case $N \geq 2$ (see Section 4 for the case $N = 1$). In particular, we will give the proofs of Lemma 2.1 and Lemma 2.3(iv). Part of the material presented in this section should be well known to experts and appears in the standard literature in the special case $N = 3$ (see e.g. [19]). However we couldn't find the corresponding formulas for the general N-dimensional case. Moreover, as already remarked in the introduction, it seems that the real part of the capacity operator has not been studied in the degree of detail which we need for our purposes.

Let $N \geq 2$, and consider the exterior Dirichlet problem for the Helmholtz equation

$$\begin{cases} \Delta w + k^2 w = 0 & \text{in } E_R := \mathbb{R}^N \backslash \overline{B_R}, \\ w = u & \text{on } S_R, \end{cases} \tag{46}$$

where $k > 0$, $R > 0$ and $u \in H^{\frac{1}{2}}(S_R, \mathbb{C})$ are given. Recall the space $H_R^{\mathbb{C}}$ defined in (8) which enforces a weak form of Sommerfeld's radiation condition. We wish to construct the unique solution $w \in H_R^{\mathbb{C}}$ explicitly in terms of a Fourier representation of the boundary datum u. For this we let Δ_S denote the Laplace-Beltrami operator on the unit sphere $S_1 \subset \mathbb{R}^N$, so that

$$\Delta w = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left(r^{N-1} \frac{\partial w}{\partial r} \right) + \frac{1}{r^2} \Delta_S w. \tag{47}$$

We recall (see e.g. [24, Corollary 2.3]) that the linear span of the spherical harmonics (i.e. restrictions to S_1 of harmonic polynomials on $\mathbb{R}^N$ with complex coefficients) is dense in $L^2(S_1, \mathbb{C})$. More precisely, denoting by $\mathcal{H}_\ell^N$ the space of spherical harmonics of degree $\ell \in \mathbb{N}_0$, then $d_\ell^N := \dim \mathcal{H}_\ell^N = \frac{(N+2\ell-2)}{(N+\ell-2)} \binom{N+\ell-2}{\ell}$ if $\ell \geq 1$, $d_0^N := \dim \mathcal{H}_0^N = 1$, and span $\bigcup_{\ell=0}^{\infty} \mathcal{H}_\ell^N$ is a dense subspace of $L^2(S_1, \mathbb{C})$. Furthermore, according to (47), every nonzero $\mathcal{Y}_\ell \in \mathcal{H}_\ell^N$ is an eigenfunction of Δ_S:

$$\Delta_S \mathcal{Y}_\ell + \ell(N + \ell - 2)\mathcal{Y}_\ell = 0 \quad \text{in } S_1 \tag{48}$$

see [11, Theorem 3.2.11]. Moreover, starting from the orthogonal bases $\{1\}$ of $\mathcal{H}_0^2$ and $\{e^{i\ell\theta}, e^{-i\ell\theta}\}$ of $\mathcal{H}_\ell^2$, $\ell \geq 1$, ($\theta = \theta(x_1, x_2) = \operatorname{sgn}(x_2) \arccos x_1$) we can inductively construct, according to [11, Lemma 3.5.3], orthogonal bases of $\mathcal{H}_\ell^N$ $\{\mathcal{Y}_1^\ell, \ldots, \mathcal{Y}_{d_\ell^N}^\ell\}$, $\ell \in \mathbb{N}_0$ for all $N \geq 2$, with the property that for each element $\mathcal{Y}$ in such a basis, the element $\overline{\mathcal{Y}}$ also belongs to this basis.

Proposition 6.1. *For $u \in H^{\frac{1}{2}}(S_R)$ given by the expansion $u(R\xi) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell^N} u_m^\ell \mathcal{Y}_m^\ell(\xi)$, $\xi \in S_1$, the problem (46) has a unique solution $w \in H_R^{\mathbb{C}}$ given by*

$$w(x) = \sum_{\ell=0}^{\infty} \left(\frac{r}{R} \right)^{-\frac{N-2}{2}} \frac{H_{\nu_\ell}^{(1)}(kr)}{H_{\nu_\ell}^{(1)}(kR)} \sum_{m=1}^{d_\ell^N} u_m^\ell \mathcal{Y}_m^\ell(\xi) \quad \textit{for } r \geq R,\ \xi \in S_1 \textit{ and } x = r\xi, \tag{49}$$

where, here and in the following, $\nu_\ell = \ell + \frac{(N-2)}{2}$ for $\ell \in \mathbb{N}_0$ and $H_\nu^{(1)}$, $H_\nu^{(2)}$ are the two Hankel functions (or Bessel functions of the third kind) of order ν (see e.g. [28, §3.6]).

Proof. We start by rescaling the space variable x to $\frac{x}{R}$. Setting $w(x) = v(\frac{x}{R})$, the problem (46) becomes

$$\begin{cases} \Delta v + (kR)^2 v = 0 & \text{in } E_1, \\ v = u_R & \text{on } S_1, \end{cases} \tag{50}$$

where $u_R(\xi) = u(R\xi)$, $\xi \in S_1$. We first remark that $w \in H_R^{\mathbb{C}}$ if and only if v belongs to the space

$$\begin{aligned} H_{1,kR}^{\mathbb{C}} := \Big\{ u \in H^1_{loc}(E_1, \mathbb{C}) \, : \frac{u}{(1+r^2)^{1/2}} \in L^2(E_1,\mathbb{C}),\ \frac{\nabla u}{(1+r^2)^{1/2}} \in L^2(E_1, \mathbb{C}^N), \\ \frac{\partial u}{\partial r} - ikRu \in L^2(E_1, \mathbb{C}) \Big\}. \end{aligned} \tag{51}$$

We will therefore seek solutions in this space, of the form $v_m^\ell(x) = f_\ell(r)\mathcal{Y}_m^\ell(\xi)$ for some $\ell \in \mathbb{N}_0$, $1 \le m \le d_l^N$. Using (47) and (48), we may rewrite (50) for functions of this form to obtain

$$0 = \{ f_\ell''(r) + \frac{(N-1)}{r} f_\ell'(r) + \left((kR)^2 - \frac{\ell(N+\ell-2)}{r^2} \right) f_\ell(r) \} \mathcal{Y}_m^\ell(\xi).$$

Setting $g_\ell(s) = (\frac{s}{kR})^{\frac{N-2}{2}} f_\ell(\frac{s}{kR})$, we find

$$g_\ell''(s) + \frac{1}{s} g_\ell'(s) + \left\{ 1 - \frac{(\ell + \frac{N-2}{2})^2}{s^2} \right\} g_\ell(s) = 0, \quad s > 0,$$

which is Bessel's equation with parameter $\nu = \ell + \frac{N-2}{2}$. Its general solution is given by

$$g(s) = AH_\nu^{(1)}(s) + BH_\nu^{(2)}(s),$$

where $A, B \in \mathbb{C}$, and $H_\nu^{(1)}$, $H_\nu^{(2)}$ are the two Hankel functions of order ν. Now, we observe that the Sommerfeld radiation condition, given as the third condition in the definition of the space $H_{1,kR}^{\mathbb{C}}$ in (51), can be rewritten in the form

$$\begin{aligned} \int_{kR}^\infty s^{-1} \big| 2As[(H_\nu^{(1)})'(s) - iH_\nu^{(1)}(s)] - A(N-2)H_\nu^{(1)}(s) \\ + 2Bs[(H_\nu^{(2)})'(s) - iH_\nu^{(2)}(s)] - B(N-2)H_\nu^{(2)}(s) \big|^2 \, ds < \infty. \end{aligned}$$

Using the recurrence formula $(H_\nu^{(p)})'(s) = \dfrac{\nu}{s} H_\nu^{(p)}(s) - H_{\nu+1}^{(p)}(s)$, $p = 1, 2$, we obtain

$$\begin{aligned} \int_{kR}^\infty s^{-1} \big| 2As[H_{\nu+1}^{(1)}(s) + iH_\nu^{(1)}(s)] + A(N-2-2\nu)H_\nu^{(1)}(s) \\ + 2Bs[H_{\nu+1}^{(2)}(s) + iH_\nu^{(2)}(s)] + B(N-2-2\nu)H_\nu^{(2)}(s) \big|^2 \, ds < \infty. \end{aligned} \tag{52}$$

According to [28, Formulas 7.2 (1) and (2)], the asymptotic behavior of $H_\nu^{(p)}(s)$, $p = 1, 2$, is given by

$$H_\nu^{(1)}(s) = \sqrt{\frac{2}{\pi s}}\, e^{i(s - \frac{2\nu+1}{4}\pi)}[1 + O(s^{-1})],\ H_\nu^{(2)}(s) = \sqrt{\frac{2}{\pi s}} e^{-i(s - \frac{2\nu+1}{4}\pi)}[1 + O(s^{-1})].$$

As a consequence,

$$|H_\nu^{(p)}(s)| = s^{-\frac{1}{2}} \{ \sqrt{\frac{2}{\pi}} + O(s^{-1}) \} \qquad \text{for } p = 1, 2,$$

and

$$s|H^{(1)}_{\nu+1}(s)+iH^{(1)}_{\nu}(s)|=O(s^{-\frac{1}{2}}), \qquad s|H^{(2)}_{\nu+1}(s)+iH^{(2)}_{\nu}(s)|=s^{\frac{1}{2}}[2\sqrt{\frac{2}{\pi}}+O(s^{-1})].$$

Thus, (52) can only be satisfied, if $B=0$. We conclude that a function in $H^{\mathbb{C}}_{1,kR}$ of the form $v^\ell_m(x)=f_\ell(r)Y^\ell_m(\xi)$ is a solution of the differential equation in (50) if and only if it can be written as

$$v^\ell_m(x)=Ar^{-\frac{(N-2)}{2}}\,H^{(1)}_{\nu_\ell}(kRr)\,\mathcal{Y}^\ell_m(\xi)$$

for some $A\in\mathbb{C}$ with $\nu_\ell=\ell+\frac{N-2}{2}$. Since $u_R(\xi)=\sum\limits_{\ell=0}^{\infty}\sum\limits_{m=1}^{d^N_\ell}u^\ell_m\mathcal{Y}^\ell_m(\xi)$, a solution of the boundary value problem (50) is thus given by

$$v(x)=\sum_{\ell=0}^{\infty}r^{-\frac{(N-2)}{2}}\frac{H^{(1)}_{\nu_\ell}(kRr)}{H^{(1)}_{\nu_\ell}(kR)}\sum_{m=1}^{d^N_\ell}u^\ell_m\mathcal{Y}^\ell_m(\xi)\quad \text{for } r\geq 1,\ \xi\in S_1 \text{ and } x=r\xi.$$

Rescaling back, we thus obtain the formula (49) for the (unique) solution $w\in H^{\mathbb{C}}_R$ of (46). $\square$

The formula (49) gives rise to the following expression for the capacity operator T_R : $H^{\frac{1}{2}}(S_R,\mathbb{C})\to H^{-\frac{1}{2}}(S_R,\mathbb{C})$ (see [19] for the case $N=3$):

$$[T_Ru](R\xi)=\frac{1}{R}\sum_{\ell=0}^{\infty}z_\ell(kR)\sum_{m=1}^{d^N_\ell}u^\ell_m\mathcal{Y}^\ell_m(\xi), \qquad \xi\in S_1 \tag{53}$$

for $u\in H^{\frac{1}{2}}(S_R,\mathbb{C})$ given by $u(R\xi)=\sum\limits_{\ell=0}^{\infty}\sum\limits_{m=1}^{d^N_\ell}u^\ell_m\mathcal{Y}^\ell_m(\xi)$, $\xi\in S_1$, where

$$z_\ell(r)=\frac{r\frac{d}{dr}H^{(1)}_{\nu_\ell}(r)}{H^{(1)}_{\nu_\ell}(r)}-\frac{N-2}{2}, \quad r>0.$$

We need some estimates for the coefficients $z_\ell(r)$. For this we recall that

$$H^{(1)}_\nu=J_\nu+iY_\nu, \quad \text{and} \quad H^{(2)}_\nu=J_\nu-iY_\nu=\overline{H^{(1)}_\nu} \qquad \text{for } \nu\in\mathbb{R},$$

where J_ν and Y_ν denote the Bessel functions of the first and second kind. Setting $G_\nu(r)=\frac{r\frac{d}{dr}H^{(1)}_\nu(r)}{H^{(1)}_\nu(r)}$, for $r>0$, $\nu\in\mathbb{R}$, and we then see that

$$\operatorname{Re}G_\nu(r)=\frac{r\operatorname{Re}(\overline{H^{(1)}_\nu(r)}\frac{d}{dr}H^{(1)}_\nu(r))}{|H^{(1)}_\nu(r)|^2}=\frac{r}{2}\frac{\frac{d}{dr}|H^{(1)}_\nu(r)|^2}{|H^{(1)}_\nu(r)|^2}=\frac{r}{2}\frac{\frac{d}{dr}(J^2_\nu(r)+Y^2_\nu(r))}{J^2_\nu(r)+Y^2_\nu(r)} \tag{54}$$

$$\operatorname{Im}G_\nu(r)=\frac{r\operatorname{Im}(\overline{H^{(1)}_\nu(r)}\frac{d}{dr}H^{(1)}_\nu(r))}{|H^{(1)}_\nu(r)|^2}=\frac{r(J_\nu(r)Y'_\nu(r)-J'_\nu(r)Y_\nu(r))}{|H^{(1)}_\nu(r)|^2}. \tag{55}$$

We need the following estimates (see [19] for the case $N=3$):

Lemma 6.2. *For $r>0$ and $\ell\geq\max\{0,\frac{3-N}{2}\}$ we have*

$$\frac{(N-1)}{2}\leq -Re\ z_\ell(r)\leq \ell+N-2 \tag{56}$$

$$0<Im\ z_\ell(r)\leq r. \tag{57}$$

In the case $N=2$, we also have $0<-Re\ z_0(r)\leq\frac{1}{2}$ and $Im\ z_0(r)>0$ for all $r>0$.

Proof. We first prove (56). According to [28, §13.74], we have

$$(i)\qquad \frac{d}{dr}(J_\nu^2(r)+Y_\nu^2(r))<0 \quad \text{for } r>0, \nu\in\mathbb{R},$$

$$(ii)\qquad \frac{d}{dr}\,r(J_\nu^2(r)+Y_\nu^2(r))\begin{cases}\le 0 & \text{if } \nu\ge\frac12\\ \ge 0 & \text{if } \nu\le\frac12,\end{cases} \qquad \text{for } r>0.$$

From (i) we obtain Re $G_\nu(r)<0$ for all $r>0$, $\nu\in\mathbb{R}$, and (ii) gives Re $G_\nu(r)\le -\frac12$ for $r>0$, $\nu\ge\frac12$. Moreover, (ii) implies Re$G_{\frac12}(r)=-\frac12$ for $r>0$. Now, using the recurrence formula $\frac{d}{dr}H_\nu^{(1)}(r)=H_{\nu-1}^{(1)}(r)-\frac{\nu}{r}H_\nu^{(1)}(r)$, we find that

$$\begin{aligned}|H_\nu^{(1)}(r)|^2(\text{Re } G_\nu(r)+\nu) &= \text{Re}\Big(r\overline{H_\nu^{(1)}(r)}\frac{d}{dr}H_\nu^{(1)}(r)+\nu|H_\nu^{(1)}(r)|^2\Big)\\ &= r\text{Re}(\overline{H_\nu^{(1)}(r)}H_{\nu-1}^{(1)}(r)) = r\text{Re}(\overline{H_{\nu-1}^{(1)}(r)}H_\nu^{(1)}(r))\\ &= -|H_{\nu-1}^{(1)}(r)|^2(\text{Re } G_{\nu-1}(r)-(\nu-1)).\end{aligned}$$

Since Re $G_\nu(r)<0$ holds for all ν, the previous formula gives Re $G_\nu(r)\ge-\nu$ for all $\nu\ge1$. Summarizing, we have shown

$$-\nu\le \text{Re } G_\nu(r)\le -\frac12,\quad r>0 \qquad \text{for } \nu\in\{\tfrac12\}\cup[1,\infty).$$

This shows (56), since $z_\ell(r)=G_{\ell+\frac{N-2}{2}}(r)-\frac{N-2}{2}$. In the case $\nu=0$, we have $-\frac12\le$ Re $G_0(r)<0$ for all $r>0$. Hence, when $N=2$, we obtain $-\frac12\le z_0(r)<0$ for all $r>0$.

We now turn to the proof of (57). Using (55) and the fact that

$$\mathcal{W}(J_\nu,Y_\nu;r):=J_\nu(r)Y_\nu'(r)-J_\nu'(r)Y_\nu(r)=\frac{2}{\pi r}\qquad \text{for } \nu\in\mathbb{R},\ r>0,$$

(see [28, §3.63 (1)]), we find that

$$\text{Im } G_\nu(r)=\frac{r\mathcal{W}(J_\nu,Y_\nu;r)}{|H_\nu^{(1)}(r)|^2}=\frac{2}{\pi|H_\nu^{(1)}(r)|^2}>0\qquad \text{for } \nu\in\mathbb{R},\ r>0.$$

Furthermore, using (ii) above, we see that

$$r|H_\nu^{(1)}(r)|^2\ge\lim_{t\to\infty}t|H_\nu^{(1)}(t)|^2=\frac{2}{\pi}\qquad \text{for } r>0, \nu\ge\frac12.$$

We thus obtain Im $G_\nu(r)\le r$ for $r>0$, $\nu\ge\frac12$. As before, (57) follows from the identity $z_\ell(r)=G_{\ell+\frac{N-2}{2}}(r)-\frac{N-2}{2}$. □

Recall that in Section 2 we have also introduced the operator

$$K_R: H^{\frac12}(S_R)\to H^{-\frac12}(S_R),\qquad K_Ru=\text{Re}[T_Ru], \tag{58}$$

where $H^{\frac12}(S_R)=H^{\frac12}(S_R,\mathbb{R})$, $H^{-\frac12}(S_R)=H^{-\frac12}(S_R,\mathbb{R})$. For $u\in H^{\frac12}(S_R)$ given by $u(R\xi)=\sum\limits_{\ell=0}^{\infty}\sum\limits_{m=1}^{d_\ell^N}u_m^\ell\mathcal{Y}_m^\ell(\xi)$ we then have

$$K_Ru=\frac1R\sum_{\ell=0}^{\infty}\text{Re}(z_\ell(kR))\sum_{m=1}^{d_\ell^N}u_m^\ell\mathcal{Y}_m^\ell, \tag{59}$$

since $\sum_{m=1}^{d_\ell^N} u_m^\ell \mathcal{Y}_m^\ell \in \mathbb{R}$ for every $\ell \in \mathbb{N}_0$. With the help of Lemma 6.2, we may therefore easily complete the

Proof of Lemma 2.1. We first note that, for $s \in \mathbb{R}$, one may define an equivalent norm on $H^s(S_R, \mathbb{C})$ by

$$\|u\|_s^2 = \sum_{\ell=0}^{\infty} (1+\ell^2)^s \sum_{m=1}^{d_\ell^N} |u_m^\ell|^2$$

for $u \in H^{\frac{1}{2}}(S_R, \mathbb{C})$ given by $u(R\xi) = \sum\limits_{\ell=0}^{\infty} \sum\limits_{m=1}^{d_\ell^N} u_m^\ell \mathcal{Y}_m^\ell(\xi)$. Hence, by (56) and (57), the capacity operator $T_R : H^{\frac{1}{2}}(S_R, \mathbb{C}) \to H^{-\frac{1}{2}}(S_R, \mathbb{C})$ is bounded, and thus K_R defined in (58) is bounded as well. In particular, the integrals on both sides of (13) exist, and by (59) and the orthogonality of spherical harmonics (corresponding to different values of ℓ) they coincide. Finally, the second inequality in (12) also follows, by orthogonality, from (59) and the estimates in Lemma 6.2. ☐

Next, we study the eigenspace X^0 of the eigenvalue problem (16) corresponding to the eigenvalue k^2, and we complete the

Proof of Lemma 2.3(iv). Recall that $u \in X^0$ if and only if u solves

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } B_R, \\ \dfrac{\partial u}{\partial \eta} = v & \text{on } S_R \end{cases} \tag{60}$$

with $v = K_R u$ on S_R. Put, as before, $\nu_\ell = \ell + \frac{N-2}{2}$ for $\ell \in \mathbb{N}_0$. In the case where for all $\ell \in \mathbb{N}_0$, $\frac{d}{dr}\left(r^{\frac{2-N}{2}} J_{\nu_\ell}(kr)\right)\Big|_{r=R} \neq 0$, the inhomogeneous Neumann problem (60) has — for given $v \in H^{-\frac{1}{2}}(S_R)$ — the unique solution

$$u(r\xi) = \sum_{\ell=0}^{\infty} \left(\frac{r}{R}\right)^{-\frac{N-2}{2}} \frac{J_{\nu_\ell}(kr)}{kJ'_{\nu_\ell}(kR) - \frac{(N-2)}{2R} J_{\nu_\ell}(kR)} \sum_{m=1}^{d_\ell^N} v_m^\ell \mathcal{Y}_m^\ell(\xi), \qquad 0 < r < R,$$

where the coefficients v_m^ℓ are determined by $v(R\xi) = \sum\limits_{\ell=0}^{\infty} \sum\limits_{m=1}^{d_\ell^N} v_m^\ell \mathcal{Y}_m^\ell(\xi)$. In particular, the restriction of u to S_R satisfies

$$u(R\xi) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell^N} u_m^\ell \mathcal{Y}_m^\ell(\xi) \qquad \text{with } u_m^\ell = \frac{J_{\nu_\ell}(kR)}{kJ'_{\nu_\ell}(kR) - \frac{(N-2)}{2R} J_{\nu_\ell}(kR)} v_m^\ell.$$

If, in addition, $v = K_R u$ on S_R, then by (59) we have

$$\begin{aligned} v_m^\ell &= \frac{1}{R} \mathrm{Re} \left(\frac{kR \frac{d}{dr} H_{\nu_\ell}^{(1)}(kR)}{H_{\nu_\ell}^{(1)}(kR)} - \frac{N-2}{2} \right) u_m^\ell \\ &= k \left(\frac{J'_{\nu_\ell}(kR) J_{\nu_\ell}(kR) + Y'_{\nu_\ell}(kR) Y_{\nu_\ell}(kR)}{J_{\nu_\ell}^2(kR) + Y_{\nu_\ell}^2(kR)} - \frac{N-2}{2kR} \right) u_m^\ell \end{aligned}$$

and therefore

$$u_m^\ell = \frac{J_{\nu_\ell}(kR)}{J'_{\nu_\ell}(kR) - \frac{(N-2)}{2kR} J_{\nu_\ell}(kR)} \left(\frac{J'_{\nu_\ell}(kR) J_{\nu_\ell}(kR) + Y'_{\nu_\ell}(kR) Y_{\nu_\ell}(kR)}{J_{\nu_\ell}^2(kR) + Y_{\nu_\ell}^2(kR)} - \frac{N-2}{2kR} \right) u_m^\ell$$

for $\ell \in \mathbb{N}_0$, $1 \le m \le d_\ell^N$. This gives for each $\ell \in \mathbb{N}_0$ the alternative

$$u_m^\ell = 0 \quad \text{for all } 1 \le m \le d_\ell^N \qquad \text{or} \qquad Y_{\nu_\ell}(kR) = 0.$$

We may now finish the proof by setting

$$\mathcal{D} := \Big\{ R > 0 \,:\, \frac{d}{dr}\Big(r^{\frac{2-N}{2}} J_{\nu_\ell}(kr)\Big)\Big|_{r=R} = 0 \text{ or } Y_{\nu_\ell}(kR) = 0 \text{ for some } \ell \in \mathbb{N}_0 \Big\}$$

Indeed, since J_{ν_ℓ} and Y_{ν_ℓ} are analytic functions on $(0,\infty)$ for every $\ell \in \mathbb{N}_0$, the set $\mathcal{D}$ is countable. Moreover, for $R \in (0,\infty) \setminus \mathcal{D}$ and $u \in X^0$ we conclude $u_m^\ell = 0$ for all $\ell \in \mathbb{N}_0$, $1 \le m \le d_\ell^N$ and therefore $u \equiv 0$. □

We close this section with a partial answer on the question asked in the end of the introduction.

Remark 6.3. In our variational framework for proving existence of solutions, the parameter $R > 0$ was always fixed. As announced in the introduction, we come back to the question whether a different choice $R' > R$ gives rise to different nontrivial solutions of (4), (6) in our approach. This is true up to countably many choices of R'. We now explain this in case $N \ge 2$, the argument in the one-dimensional case is similar but easier. Let $R' > R > 0$ be given, and suppose that u_1 and u_2 are solutions of the linear Helmholtz equation in E_R with the following properties:

(i) $u_1 \equiv u_2$ on $E_{R'}$;
(ii) u_1 coincides with the real part of a solution w_1 on E_R satisfying the Sommerfeld radiation condition (5);
(iii) u_2 coincides with the real part of a solution w_2 on $E_{R'}$ satisfying (5);
(iv) $u_1|_{S_R} \equiv w_1|_{S_R}$ and $u_2|_{S_{R'}} \equiv w_2|_{S_{R'}}$.

Since $w := w_1 - w_2$ also satisfies the linear Helmholtz equation on $E_{R'}$ together with (5) and $\mathrm{Re}(w) = u_1 - u_2 \equiv 0$ on $E_{R'}$, it is easy to see that $w \equiv 0$ and hence $w_1 = w_2$ on $E_{R'}$. By (i) and (iv) this implies that $\mathrm{Im}(w_1|_{S_{R'}}) \equiv 0$. Writing again

$$u_1(R\xi) = w_1(R\xi) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell^N} u_m^\ell \mathcal{Y}_m^\ell(\xi), \qquad \xi \in S_1,$$

we have, by Proposition 6.1

$$w_1(R'\xi) = \sum_{\ell=0}^{\infty} \left(\frac{R'}{R}\right)^{-\frac{(N-2)}{2}} \frac{H_{\nu_\ell}^{(1)}(kR')}{H_{\nu_\ell}^{(1)}(kR)} \sum_{m=1}^{d_\ell^N} u_m^\ell \mathcal{Y}_m^\ell(\xi), \qquad \xi \in S_1,$$

so that $\mathrm{Im}(w_1|_{S_{R'}}) \equiv 0$ is equivalent to the following statement: For all $\ell \in \mathbb{N}_0$ we have $u_m^\ell = 0$ for $m = 1, \dots, d_\ell^N$ or

$$\mathrm{Im}\Big(\frac{H_{\nu_\ell}^{(1)}(kR')}{H_{\nu_\ell}^{(1)}(kR)}\Big) = 0, \qquad \text{i.e.,} \qquad J_{\nu_\ell}(kR')Y_{\nu_\ell}(kR) = J_{\nu_\ell}(kR)Y_{\nu_\ell}(kR').$$

Since the functions J_{ν_ℓ} and Y_{ν_ℓ} are real analytic, the latter can only happen for countably many $R' > R$. Consequently, there exists a countable set $\mathcal{D}_R \subset (R,\infty)$ such that, if $R' \notin \mathcal{D}_R$, we have $u_m^\ell = 0$ for $\ell \in \mathbb{N}_0$ and $m = 1, \dots, d_\ell^N$, i.e. $u_1|_{S_R} \equiv 0$ and therefore $u_1 \equiv 0$. Hence, choosing $R' > R$ such that $R' \notin \mathcal{D}_R$ in place of R, we see that our approach yields different nontrivial solutions.

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